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UNIVERSAL SOLUTIONS FOR FIBER-REINFORCED INCOMPRESSIBLE ISOTROPIC ELASTIC MATERIALS

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Abstract—Universal solutions for fiber-reinforced incompressible isotropic elastic materials under large elastic deformations are obtained by inverse methods.

The following deformations are investigated: bending and shearing of a rectangular block; straightening and shearing of a sector of a circular tube; inflation, eversion, extension, bending and shearing of a sector of a circular tube.

A discussion is made about the strengthening of the material due to the reinforcement, the sign of the tension in the fibers and the deformed configuration of the fibers.

1. INTRODUCTION

IN RECENT years, the mechanics of composite materials has been studied extensively in the framework of linear elasticity. However, for materials which undergo large elastic deformations, non-linear elasticity models should be used.

The general theory of hyperelastic materials with internal constraints has been developed by Ericksen and Rivlin [1] and an exposition of this theory is also contained in Green and Adkins [2]. Truesdell and Noll [3] presented this theory for general elastic materials. Rivlin and Adkins [4] and Adkins [5–7] solved several problems of large elastic deformations of incompressible hyperelastic materials reinforced by discrete layers of inextensible cords. Ericksen and Rivlin [1] have studied large elastic deformations of hyperelastic anisotropic materials. Pipkin and Rogers [8] investigated plane deformations of incompressible fiber-reinforced materials.

Here the case of an incompressible isotropic elastic material reinforced by a system of thin, flexible, inextensible fibers filling it continuously and completely is considered. The reinforced body is treated as a material subject to internal constraints according to the reference [3]. It is attempted to find so-called universal solutions, which are present independent of the details of the constitutive equation of the material.

Ericksen [9] proved that all possible universal solutions for incompressible isotropic elastic materials can be classified into six families. However, Singh and Pipkin [10] later found one more such family.

It is of interest to attack the problem of determining all possible universal solutions of an incompressible isotropic elastic material reinforced by a system of fibers. Here, however, we deal only with the problem of constructing by inverse methods certain universal solutions which, we expect, will contribute to the understanding of the behavior of fiber-reinforced composite materials under large elastic deformations.

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To this end the following deformations are investigated in detail: bending and shearing of a rectangular block; straightening and shearing of a sector of a circular tube; inflation eversion, extension, torsion, bending and shearing of a sector of a circular tube.

The engineering objective of introducing systems of fibers into a material is to generate additional strength in the composite material. The influence of the reinforcement on the various deformations considered here is discussed and the cases where the contribution is significant are recognized. A measure of the significance of the reinforcement for a given deformation is the ratio of the maximum extra stress to the fiber tension.

It is found that, depending on the deformation and the arrangement of fibers, the contribution of the reinforcement can be significant as e.g. in bending and extension of a sector of a circular tube reinforced with fibers along the R direction, or weak as e.g. in torsion of a circular tube reinforced with concentric rings of fibers. The two extremes of rendering the body rigid (as e.g. in inflation of a circular tube reinforced with concentric rings), or not strengthening it at all (as e.g. in extension of a circular tube reinforced by two sets of fibers of helical path) also appear.

Since the fibers are thin and flexible, their tension must be positive to avoid possible buckling. Thus an investigation of the sign of the tension in the fibers is made for the various deformations. It appears that in the case of torsion of a circular tube reinforced with concentric rings or radially as well as in the case of shearing of a sector of a circular tube radially reinforced the fiber tension is always positive. In all the other cases the reinforcement can be positive under appropriate conditions on the various constants of the deformation and the response coefficients.

2. EQUILIBRIUM AND CONSTITUTIVE EQUATIONS

In this paper we deal with deformations of rectangular blocks and circular tubes. In accordance, Cartesian and cylindrical coordinate systems are used in our analysis. In the latter case the equilibrium equations, in the absence of body forces, take the form

$$
\begin{aligned}
t^r_{r,r}+t^\theta_{r,\theta}+t^z_{r,z}+\frac{1}{r}(t^r_r-t^\theta_\theta) &= 0,\\
t^r_{\theta,r}+t^\theta_{\theta,\theta}+t^r_{\theta,z}+rt^\theta_r &= 0,\\
t^r_{z,r}+t^\theta_{z,\theta}+t^z_{z,z}+\frac{1}{r}t^r_z &= 0.
\end{aligned}
\tag{2.1}
$$

A static configuration of the body is determined by the invertible transformation

$$x^i = x^i(X^\alpha).* \tag{2.2}$$

The deformation gradient F^i_α is defined as

$$F^i_\alpha = x^i{}_{,\alpha} \tag{2.3}$$

* Here and throughout, Greek indices refer to material coordinates X^α, Latin to the spatial coordinates x^i, summation convention is used and , indicates differentiation.

and the right and left Cauchy–Green deformation tensors are given by

$$C_{\alpha\beta} = g_{ij}F^i_{\alpha}F^j_{\beta}, \tag{2.4}$$

$$B^i_j = g_{jk}G^{\alpha\beta}F^k_{\alpha}F^i_{\beta}. \tag{2.5}$$

Here $G^{\alpha\beta}$ and g_{ij} are the metric tensors in the reference and present configuration, respectively.

The constitutive equation for a homogeneous incompressible fiber-reinforced isotropic elastic material takes the form (e.g. [3, Sections 30, 49])

$$t^i_j = -p\delta^i_j + \beta_1 B^i_j + \beta_{-1}(B^{-1})^i_j - N^i_j, \tag{2.6}$$

with

$$N^i_j = qg_{jk}F^k_{\alpha}F^i_{\beta}e^{\alpha}e^{\beta}. \tag{2.7}$$

The hydrostatic pressure p in (2.6) and the scalar factor q in (2.7) are, in general, scalar functions of X^{α}. The response coefficients β_1 and β_{-1} in (2.6) are functions of the first two principal invariants of the matrix B^i_j and they satisfy the inequalities [3]

$$\beta_1 > 0, \qquad \beta_{-1} \leq 0. \tag{2.8}$$

Finally, in (2.7), e is a unit vector field on the reference configuration tangential at every point to the fiber passing from this point.

The incompressibility and inextensibility conditions are

$$\det[B^i_j] = 1 \tag{2.9}$$

and

$$C_{\alpha\beta}e^{\alpha}e^{\beta} = 1 \tag{2.10}$$

respectively [3].

In elasticity, it is customary to use physical rather than tensor components of stress. These are given in terms of the tensor components as follows for a cylindrical system of co-ordinates:

$$t_{\langle rr\rangle} = t^r_r, \qquad t_{\langle r\theta\rangle} = rt^{\theta}_r = \frac{1}{r}t^r_{\theta}, \qquad t_{\langle rz\rangle} = t^r_z = t^z_r,$$

$$t_{\langle \theta\theta\rangle} = t^{\theta}_{\theta}, \qquad t_{\langle \theta z\rangle} = rt^{\theta}_z = \frac{1}{r}t^z_{\theta}, \qquad t_{\langle zz\rangle} = t^z_z.$$

3. BENDING AND SHEARING OF A RECTANGULAR BLOCK REINFORCED WITH FIBERS ALONG THE Z DIRECTION

The reference configuration of the body is referred to a Cartesian co-ordinate system X, Y, Z, while the present configuration to a cylindrical system r, θ, z.

Deformations of the following type are considered:

$$r = r(X), \qquad \theta = AY, \qquad z = BZ - CY, \tag{3.1}$$

where the constants A, B and C correspond to bending, stretching and shearing, respectively.

The block in its natural state is bounded by the planes $X = X_1$, $X = X_2$ $(X_2 > X_1)$, $Y = \pm b$ and $Z = \pm \alpha$, while in the present configuration the strained block is bounded by the curved surfaces $r = r_1 = r(X_1)$, $r = r_2 = r(X_2)$, the planes $\theta = \pm\theta_0 = \pm Ab$ and the helicoidal surfaces $z = BZ - CY = \pm B\alpha - (C/A)\theta$.

A simple computation yields the Cauchy–Green tensors:

$$[C_{\alpha\beta}] = \begin{bmatrix} r_X^2 & 0 & 0 \\ 0 & C^2 + r^2A^2 & -BC \\ 0 & -BC & B^2 \end{bmatrix}, \tag{3.2}$$

$$[B^i_j] = \begin{bmatrix} r_X^2 & 0 & 0 \\ 0 & r^2A^2 & -AC \\ 0 & -r^2AC & B^2 + C^2 \end{bmatrix}, \tag{3.3}$$

$$[(B^{-1})^i_j] = \begin{bmatrix} \dfrac{1}{r_X^2} & 0 & 0 \\ 0 & \dfrac{B^2 + C^2}{r^2A^2B^2} & \dfrac{C}{r^2AB^2} \\ 0 & \dfrac{C}{AB^2} & \dfrac{1}{B^2} \end{bmatrix}. \tag{3.4}$$

The incompressibility condition (2.9) becomes

$$r^2 = \frac{2}{AB}X + N, \tag{3.5}$$

where N is a constant of integration.

Reinforcement along the Z direction is considered only,* in which case the inextensibility condition (2.10) yields

$$B^2 = 1. \tag{3.6}$$

Thus (3.1) takes the form

$$r = \sqrt{\left(\frac{2}{A}X + N\right)}, \qquad \theta = AY, \qquad z = Z - CY, \tag{3.7}$$

with

$$\frac{2}{A}X + N > 0.$$

The constitutive equation (2.6) reduces to

$$t^i_j = -p\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \beta_1[B^i_j] + \beta_{-1}[(B^{-1})^i_j] - q\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{3.8}$$

where p and q are, in general, functions of r, θ, z, while β_1 and β_{-1} are functions of r only.

* Reinforcement along the X or Y directions is incompatible with the deformation (3.1).

The equilibrium equations (2.1) become

$$
\begin{gathered}
-\frac{\partial p}{\partial r}+\frac{\partial}{\partial r}\left(\beta_1\frac{1}{A^2r^2}+\beta_{-1}A^2r^2\right)+\frac{1}{r}\left[\beta_1\left(\frac{1}{A^2r^2}-r^2A^2\right)+\beta_{-1}\left(A^2r^2-\frac{1+C^2}{A^2r^2}\right)\right]=0 \\
-\frac{\partial p}{\partial\theta}=0,\qquad -\frac{\partial p}{\partial z}-\frac{\partial q}{\partial z}=0.
\end{gathered}
\tag{3.9}
$$

Integrating $(3.9)_1$ and using the condition $[t_r^r]_{r=r_1}=0$, it is found that

$$
p(r)=\beta_1\frac{1}{A^2r^2}+\beta_{-1}r^2A^2+\int_{r_1}^{r}\frac{1}{r}\left[\beta_1\left(\frac{1}{r^2A^2}-r^2A^2\right)+\beta_{-1}\left(A^2r^2-\frac{1+C^2}{A^2r^2}\right)\right]\mathrm{d}r. \tag{3.10}
$$

The imposed condition that the z component P_z of the traction on the helicoidal surface $z=Z-CY$ vanishes, yields

$$
q(r)=-p(r)+\beta_1+\beta_{-1}\left(1+\frac{C^2}{A^2r^2}\right). \tag{3.11}
$$

The above expressions for p and q satisfy $(3.9)_2$ and $(3.9)_3$.

The constants A, C and N are computed by imposing the conditions $[t_r^r]_{r=r_2}=0$, $F_\theta=0$ and $S_\theta=0$, where F_θ is the θ component of the force on the helicoidal surface and S_θ is the shearing force on the planes $\theta=\pm\theta_0$. Existence of solution depends, of course, on the nature of the β_1 and β_{-1}.

The block is kept in its deformed state by exerting the tractions P_θ on the helicoidal surface and $t_{\langle\theta\theta\rangle}$ and $t_{\langle\theta z\rangle}$ on the planes $\theta=\pm\theta_0$.

The nonzero components of stress are

$$
\begin{aligned}
t_{\langle rr\rangle}&=-p(r)+\beta_1\frac{1}{A^2r^2}+\beta_{-1}A^2r^2,\\
t_{\langle\theta\theta\rangle}&=-p(r)+\beta_1A^2r^2+\beta_{-1}\frac{1+C^2}{A^2r^2},\\
t_{\langle zz\rangle}&=\beta_1C^2-\beta_{-1}\frac{C^2}{A^2r^2},\\
t_{\langle\theta z\rangle}&=\frac{1}{r}\left(-\beta_1r^2AC+\beta_{-1}\frac{C}{A}\right).
\end{aligned}
$$

The fiber tension which equals q and the maximum extra stress $t_{\langle\theta\theta\rangle}$ are of the same order of magnitude with respect to the constants A and C and hence the contribution of the reinforcement is significant here. The fiber tension q can be positive under appropriate conditions. The fibers in the strained state remain along the Z direction.

4. STRAIGHTENING AND SHEARING OF A SECTOR OF A CIRCULAR TUBE REINFORCED WITH FIBERS ALONG THE Z DIRECTION

The reference configuration of the body is referred to a cylindrical co-ordinate system R, Θ, Z, while the present configuration to a Cartesian system x, y, z.

Deformations of the following type are considered:

$$x = x(R), \qquad y = A\Theta, \qquad z = BZ + C\Theta, \tag{4.1}$$

where the constants A, B and C correspond to straightening, stretching and shearing, respectively.

The sector of the circular tube in its natural state is bounded by the curved surfaces $R = R_1$, $R = R_2 (R_2 > R_1)$ and the planes $\Theta = \pm\Theta_0$ and $Z = \pm\alpha$, while in the present configuration the strained block is bounded by the planes $x = x_1 = x(R_1)$, $x = x_2 = x(R_2)$, $y = \pm b = \pm A\Theta_0$ and $z = BZ + C\Theta = \pm B\alpha + (C/A)y$.

The method of solution is very similar to that employed in the previous section and therefore only the results of our analysis will be presented.

The deformation is

$$x = \frac{1}{2A}R^2 + N, \qquad y = A\Theta, \qquad z = Z + C\Theta \tag{4.2}$$

and the fiber tension and the nonzero stresses are given by

$$\begin{aligned} q(x) &= -\beta_1\left(\frac{R^2}{A^2} - 1\right) - \beta_{-1}\left(\frac{A^2}{R^2} - \frac{C^2}{A^2} - 1\right), \\ t_{\langle yy\rangle} &= \beta_1\left(\frac{A^2}{R^2} - \frac{R^2}{A^2}\right) + \beta_{-1}\left(\frac{R^2 + C^2}{A^2} - \frac{A^2}{R^2}\right), \\ t_{\langle yz\rangle} &= \beta_1\frac{AC}{R^2} - \beta_{-1}\frac{C}{A}, \\ t_{\langle zz\rangle} &= \beta_1\frac{C^2}{R^2} - \beta_{-1}\frac{C^2}{A^2}. \end{aligned}$$

The constants A, C and N are computed by imposing the conditions $N_y = 0$, $S_y = 0$ and $F_y = 0$, where N_y and S_y are the normal and the shearing force on the planes $y = \pm b$ and F_y is the y component of the force on the plane $z = Z + C\Theta$.

The body is kept in its deformed state by the tractions P_y on the plane $z = Z + C\Theta$ and the $t_{\langle yy\rangle}$ and $t_{\langle zy\rangle}$ on the planes $y = \pm b$.

The contribution of the reinforcement in this deformation is significant and the fiber tension is positive provided that $R_2 < A^2/\sqrt{(A^2 + C^2)}$. The fibers in the strained state remain straight along the Z direction.

5. INFLATION EVERSION EXTENSION TORSION BENDING AND SHEARING OF A SECTOR OF A CIRCULAR TUBE

Both the reference and the present configuration of the body are referred to cylindrical co-ordinate systems R, Θ, Z and r, θ, z, respectively.

Deformations of the following type are considered:

$$r = r(R), \qquad \theta = C\Theta + DZ, \qquad z = E\Theta + FZ, \tag{5.1}$$

where the constants C, D, E and F correspond to bending, torsion, shearing and extension, respectively. Inflation and eversion are described by the relation $r = r(R)$ in (5.1).

The body in its natural state is bounded by the curved surfaces $R = R_1$, $R = R_2 (R_2 > R_1)$ and the planes $\Theta = \pm\Theta_0$ and $Z = \pm L$, while in the present configuration the strained body is bounded by the curved surfaces $r = r_1 = r(R_1)$, $r = r_2 = r(R_2)$ and the helicoidal surfaces $\theta = C\Theta + DZ = (\pm C \mp DE/F)\Theta_0 + (D/F)z$ and $z = E\Theta + FZ = (\pm F \mp DE/C)L$.

A simple computation yields the tensors $C_{\alpha\beta}$, B^i_j and $(B^{-1})^i_j$.

The incompressibility condition (2.9) takes the form

$$r = \sqrt{(AR^2+B)}, \qquad A(CF-DE) = 1, \tag{5.2}$$

where A and B are constants such that $AR^2 + B > 0$.

Deformations (5.1) are studied for the following types of fiber-reinforcement:

(I) *Fibers along the R direction*

The inextensibility condition (2.10) takes the form

$$r = R + K, \tag{5.3}$$

where K is a constant.

In view of (5.2) and (5.3), (5.1) yield

$$r = R, \qquad \theta = C\Theta + DZ, \qquad z = E\Theta + FZ, \qquad CF - DE = 1. \tag{5.4}$$

The equilibrium equations (2.1) take the form

$$\begin{gathered} -\frac{\partial p}{\partial r} + \frac{\partial}{\partial r}(\beta_1 + \beta_{-1}) - \frac{\partial q}{\partial r} - \frac{q}{r} + \frac{1}{r}\left[\beta_1(1 - C^2 - r^2D^2) + \beta_{-1}\left(1 - \frac{R^2F^2 + E^2}{r^2}\right)\right] = 0, \\ \frac{\partial p}{\partial \theta} = 0, \qquad \frac{\partial p}{\partial z} = 0. \end{gathered} \tag{5.5}$$

Imposing the condition $P_z = 0$, where P_z is the z component of the traction on the helicoidal surface $z = E\Theta + FZ$, we compute

$$p(r) = \beta_1\left(F^2 - \frac{EDF}{C}\right) + \beta_{-1}\left(C^2 + R^2D^2 + \frac{E^2}{r^2} + \frac{EDFR^2}{Cr^2}\right) \tag{5.6}$$

and thus $(5.5)_1$, after an integration and use of the boundary condition $[t^r_r]_{r=r_2} = 0$, yields

$$q(r) = \frac{1}{r}\int_r^{r_2} \phi(r)\,\mathrm{d}r + \frac{r_2}{r}[\beta_1 + \beta_{-1}]_{r=r_2} - \frac{r_2}{r}p(r_2) \tag{5.7}$$

with

$$\phi(r) = r\frac{\partial p}{\partial r} - r\frac{\partial}{\partial r}(\beta_1 + \beta_{-1}) - \left[\beta_1(1 - C^2 - r^2D^2) + \beta_{-1}\left(1 - \frac{R^2F^2 + E^2}{r^2}\right)\right]. \tag{5.8}$$

More specifically, for each particular type of deformation one draws the following conclusions.

(a) *Inflation* ($C = F = 1$, $D = E = 0$), *Eversion* ($C = 1$, $D = E = 0$, $A < 0$, $F < 0$) and *Extension* ($C = 1$, $D = E = 0$) are impossible.*

(b) *Torsion* ($C = F = 1$, $E = 0$).

Equations (5.4) reduce to

$$r = R, \qquad \theta = \Theta + DZ, \qquad z = Z. \tag{5.9}$$

* Actually, this reinforcement renders the body rigid for inflation or extension, while it renders eversion impossible.

The fiber tension which equals q is

$$q(r) = D^2\left[\frac{1}{r}\int_r^{r_2}\left[r\frac{\partial}{\partial r}(\beta_{-1}r^2)+\beta_1 r^2\right]\mathrm{d}r-\frac{r_2^3}{r}(\beta_{-1})_{r=r_2}\right].$$

The torsion is produced by an assigned torque M_z and an internal pressure P computed from the condition $[t_r^r]_{r=r_1} = -P$. The expression of the torque M_z due to the stress $t_{\langle\theta z\rangle}$ serves to compute the constant D.

The nonzero components of the stress are

$$t_{\langle rr\rangle} = -\beta_{-1}r^2D^2-q(r),$$
$$t_{\langle\theta\theta\rangle} = D^2r^2(\beta_1-\beta_{-1}),$$
$$t_{\langle\theta z\rangle} = rD(\beta_1-\beta_{-1}).$$

The maximum extra stress $t_{\langle\theta z\rangle}$ is found to be proportional to D, while the fiber tension q proportional to D^2 and hence the contribution of the reinforcement is weak for small values of D. The fiber tension q can be positive under appropriate conditions on the response coefficients β_1 and β_{-1}; however, for the special case of the Neo–Hookean solid ($\beta_1 = \mu > 0$, $\beta_{-1} = 0$) q is always positive. The fibers in the strained state remain at the same level but experience a rotation DZ.

(c) *Bending and extension* ($D = E = 0$).

Equations (5.4) reduce to

$$r = R, \qquad \theta = C\Theta, \qquad z = \frac{1}{C}Z. \tag{5.10}$$

The constant C is computed by imposing the condition $[t_r^r]_{r=r_1} = 0$ and the deformation is produced by a bending moment M_θ, which is due to the stress $t_{\langle\theta\theta\rangle}$ and acts on the planes $\theta = \pm C\Theta_0$.

The fiber tension which equals q and the nonzero stresses reduce to

$$q(r) = -\beta_1\left[C^2-1+\frac{r_2}{r}\left(\frac{1}{C^2}-C^2\right)\right]-\beta_{-1}\left[\frac{1}{C^2}-1+\frac{r_2}{r}\left(C^2-\frac{1}{C^2}\right)\right]$$

$$t_{\langle rr\rangle} = \beta_1\left(1-\frac{1}{C^2}\right)+\beta_{-1}(1-C^2)-q(r),$$

$$t_{\langle\theta\theta\rangle} = (\beta_1-\beta_{-1})\left(C^2-\frac{1}{C^2}\right).$$

The fiber tension q is positive provided that $C > 1$. The maximum extra stress $t_{\langle\theta\theta\rangle}$ and the fiber tension q are of the same order of magnitude with respect to C and hence the contribution of the reinforcement is significant. The fibers in the strained state remain radial straight lines but they change level and experience a rotation $(C-1)\Theta$.

(d) *Shearing* ($C = F = 1$, $D = 0$)

Equations (5.4) reduce to

$$r = R, \qquad \theta = \Theta, \qquad z = E\Theta+Z. \tag{5.11}$$

The deformation is produced by an assigned traction P_θ on the helicoidal surface $z = E\Theta+Z$ and a traction $t_{\langle\theta z\rangle}$ on the planes $\theta = \pm\Theta_0$.

The constant E is computed from the equation

$$P_\theta = \frac{rE}{\sqrt{(r^2+E^2)}}(\beta_1-\beta_{-1}).$$

The fiber tension which equals q and the nonzero stresses are

$$q(r) = -\beta_{-1}\frac{E^2}{r^2}, \qquad t_{\langle zz\rangle} = \frac{E^2}{r^2}(\beta_1-\beta_{-1}), \qquad t_{\langle \theta z\rangle} = \frac{1}{r}E(\beta_1-\beta_{-1}).$$

The maximum extra stress $t_{\langle zz\rangle}$ and the fiber tension q are of the same order of magnitude with respect to E and this case the contribution of the reinforcement is significant; however, for small values of E, the maximum extra stress is $t_{\langle \theta z\rangle}$ and the reinforcement is weak. The sign of the fiber tension q is always positive on account of (2.8). The fibers in the strained state change their level by $E\Theta$.

(II) *Fibers along the* Θ *direction*

Following the method used in the previous case we conclude that the deformation is

$$r = \frac{1}{C}\sqrt{(R^2-E^2)}, \qquad \theta = C\Theta + DZ, \qquad z = E\Theta + FZ, \tag{5.12}$$

with

$$CF - DE = C^2, \qquad R^2 - E^2 > 0.$$

Using the condition $t_r^r = 0$,* the hydrostatic pressure p and the fiber tension which is q apart from a positive scalar factor take the form

$$p(r) = \beta_1\frac{R^2}{C^4r^2} + \beta_{-1}\frac{C^4r^2}{R^2}, \tag{5.13}$$

$$q(r) = \beta_1\left(-\frac{R^4}{C^6r^4} + \frac{D^2R^2}{C^2} + 1\right) - \beta_{-1}\left(C^2 - \frac{R^4F^2}{C^6r^4} - \frac{E^2R^2}{C^6r^4}\right). \tag{5.14}$$

More specifically, for each particular type of deformation, one draws the following conclusions.

(a) As before, *inflation, extension* and *eversion* are impossible.

(b) *Torsion* ($C = F = 1$, $E = 0$)

Equations (5.12) reduce to

$$r = R, \qquad \theta = \Theta + DZ, \qquad z = Z. \tag{5.15}$$

The deformation is produced by an assigned torque M_z and an axial force due to the stress $t_{\langle zz\rangle}$.

The constant D is computed from the expression for the torque M_z due to the stress $t_{\langle \theta z\rangle}$.

The fiber tension which equals q and the nonzero stresses are given by

$$q(r) = \beta_1 D^2R^2, \qquad t_{\langle \theta z\rangle} = rD(\beta_1-\beta_{-1}), \qquad t_{\langle zz\rangle} = \beta_{-1}D^2r^2.$$

* This condition implies that $t_\theta^\theta = 0$.

The reinforcement in this case strengthens the material slightly for small values of D. The sign of q in this case is always positive. The fibers shaped as concentric rings experience only a rigid rotation DZ during the deformation.

(c) *Bending and extension* ($D = E = 0$)

Equations (5.12) reduce to

$$r = \frac{1}{C}R, \qquad \theta = C\Theta, \qquad z = CZ. \tag{5.16}$$

The deformation is produced by an assigned traction $t_{\langle zz \rangle}$ and the constant C is computed from the following expression for $t_{\langle zz \rangle}$.

$$t_{\langle zz \rangle} = (\beta_1 - \beta_{-1})\left(C^2 - \frac{1}{C^2}\right).$$

The $t_{\langle zz \rangle}$ is the only nonzero stress here; the fiber tension which equals q is given by

$$q = -\beta_1\left(\frac{1}{C^2} - 1\right) - \beta_{-1}(C^2 - 1)$$

and is positive for $C > 1$. The contribution of the reinforcement is here significant.

The undeformed fibers shaped as circular arcs change level and curvature during deformation.

(d) *Shearing* ($C = F = 1, D = 0$)

Equations (5.12) reduce to

$$r = \sqrt{(R^2 - E^2)}, \qquad \theta = \Theta, \qquad z = Z + E\Theta, \qquad R - E > 0. \tag{5.17}$$

The constant E is computed by imposing the condition $F_\theta = 0$, where F_θ is the θ component of force on the helicoidal surface $z = Z + E\Theta$. The deformation is produced by the tractions P_θ and P_z on the helicoidal surface and the traction $t_{\langle \theta z \rangle}$ on the planes $\theta = \pm\Theta_0$.

The scalar factor q and the nonzero stresses are given by

$$q(r) = \beta_1\left(1 - \frac{R^4}{r^4}\right) - \beta_{-1}\left(1 - \frac{R^4}{r^4} - \frac{R^2}{r^4}E^2\right),$$

$$t_{\langle zz \rangle} = \beta_1\left(1 + \frac{E^2}{R^2} - \frac{R^2}{r^2}\right) + \beta_{-1}\left(1 - \frac{r^2}{R^2}\right),$$

$$t_{\langle \theta z \rangle} = \frac{1}{r}\left(\beta_1 \frac{r^2}{R^2} - \beta_{-1}\right)E + \frac{r}{R^2}Eq(r).$$

The contribution of the reinforcement is significant here and q can be positive under appropriate conditions. The fibers, originally shaped as circular arcs, change curvature and take helicoidal shape.

(III) *Fibers along the Z direction*

Following the method used in the first case we conclude that the deformation is

$$r = \sqrt{(AR^2 + B)}, \qquad \theta = C\Theta, \qquad z = E\Theta \pm Z, \qquad AC = \pm 1, \qquad AR^2 + B > 0 \tag{5.18}$$

with the minus sign corresponding to eversion. The hydrostatic pressure p and the fiber tension which equals q are given by

$$p(r) = \beta_1 \frac{A^2R^2}{r^2} + \beta_{-1}\frac{r^2}{A^2R^2} - \int_r^{r_2} \frac{1}{r}\phi(r)\,\mathrm{d}r \tag{5.19}$$

with

$$\phi(r) = \beta_1\left(\frac{A^2R^2}{r^2} - \frac{C^2r^2}{R^2}\right) + \beta_{-1}\left[\frac{r^2}{A^2R^2} - \frac{A^2(R^2+E^2)}{r^2}\right],$$

$$q(r) = -p(r) + \beta_1 + \beta_{-1}\left(\frac{A^2E^2}{r^2} + C^2A^2\right). \tag{5.20}$$

More specifically, for each particular type of deformation, one draws the following conclusions.

(a) *Inflation* ($C = F = 1$, $D = E = 0$)

Equations (5.18) reduce to

$$r = \sqrt{(R^2+B)}, \qquad \theta = \Theta, \qquad z = Z, \qquad R^2 + B > 0. \tag{5.21}$$

The deformation is produced by an assigned internal pressure P, for a complete tube, and the constant B is computed from the condition $[t_r^r]_{r=r_1} = -P$.

The fiber tension which equals q and the nonzero stresses are

$$\begin{aligned} q(r) &= -p(r) + \beta_1 + \beta_{-1}, \\ t_{\langle rr\rangle} &= -p(r) + \beta_1\frac{R^2}{r^2} + \beta_{-1}\frac{r^2}{R^2}, \\ t_{\langle\theta\theta\rangle} &= -p(r) + \beta_1\frac{r^2}{R^2} + \beta_{-1}\frac{R^2}{r^2}. \end{aligned}$$

The contribution of the reinforcement is significant in this case and q can be positive under appropriate conditions. The fibers in the strained state remain vertical and they move only in the radial direction.

(b) *Eversion* ($C = 1$, $D = E = 0$, $A < 0$, $F < 0$)

Equations (5.18) reduce to

$$r = \sqrt{(-R^2+B)}, \qquad \theta = \Theta, \qquad z = -Z, \qquad B - R^2 > 0. \tag{5.22}$$

The complete tube is turned inside out and it is free of tractions after the deformation has occurred.

Imposing the condition $[t_r^r]_{r=r_1} = 0$, the constant B can be computed. The stresses have the same form as in the case of inflation and remarks on the character of the reinforcement analogous to those stated before for inflation hold here also.

(c) *Extension* and *torsion* are impossible here.

(d) *Bending* ($D = E = 0$)

Equations (5.18) reduce to

$$r = \sqrt{\left(\frac{1}{C}R^2 + B\right)}, \qquad \theta = C\Theta, \qquad z = Z, \qquad \frac{R^2}{C} + B > 0. \tag{5.23}$$

The deformation is produced by an assigned bending moment M_θ. The constants C and B are computed by imposing the condition $[t_r^r]_{r=r_1} = 0$ and using the expression of M_θ in terms of $t_{\langle\theta\theta\rangle}$.

The fiber tension which equals q and the nonzero stresses are

$$
\begin{aligned}
q(r) &= -p(r)+\beta_1+\beta_{-1},\\
t_{\langle rr\rangle} &= -p(r)+\beta_1\frac{R^2}{r^2C^2}+\beta_{-1}\frac{r^2C^2}{R^2},\\
t_{\langle\theta\theta\rangle} &= -p(r)+\beta_1\frac{C^2r^2}{R^2}+\beta_{-1}\frac{R^2}{r^2C^2}.
\end{aligned}
$$

The contribution of the reinforcement is here significant and the q can be positive under appropriate conditions. The fibers remain vertical but move in the radial direction as well as along the circumference.

(e) *Shearing* $(C = F = 1, D = 0)$

Equations (5.18) reduce to

$$r = \sqrt{(R^2+B)}, \qquad \theta = \Theta, \qquad z = Z+E\Theta, \qquad R^2 + B > 0. \tag{5.24}$$

The constants E and B are computed by imposing the conditions $[t_r^r]_{r=r_1} = 0$ and $F_\theta = 0$. The deformation is then produced by the tractions P_θ on the helicoidal surface and $t_{\langle\theta\theta\rangle}$ and $t_{\langle\theta z\rangle}$ on the planes $\theta = \pm\Theta_0$.

The fiber tension q and the nonzero stresses are

$$
\begin{aligned}
q(r) &= -p(r)+\beta_1+\beta_{-1}\left(1+\frac{E^2}{r^2}\right),\\
t_{\langle rr\rangle} &= -p(r)+\beta_1\frac{R^2}{r^2}+\beta_{-1}\frac{r^2}{R^2},\\
t_{\langle\theta\theta\rangle} &= -p(r)+\beta_1\frac{r^2}{R^2}+\beta_{-1}\frac{R^2+E^2}{r^2},\\
t_{\langle\theta z\rangle} &= \beta_1\frac{r}{R^2}E-\frac{1}{r}\beta_{-1}E.
\end{aligned}
$$

The contribution of the reinforcement is significant and q can be positive under appropriate conditions. The fibers move in the radial and vertical direction.

(IV) *Two sets of fibers of helical path symmetrically inclined at an angle α to the Z direction*

The two sets of fibers form angles $+\alpha$ and $-\alpha$ with the Z direction and the inextensibility condition (2.10), with $e = \{0, \pm(\sin\alpha/R), \cos\alpha\}$, takes the form

$$
\begin{gathered}
C^2r^2\sin^2\alpha+F^2R^2\cos^2\alpha = R^2,\\
D = E = 0,
\end{gathered}
\tag{5.25}
$$

for both sets of fibers.

Equations (5.1) yield

$$r = \lambda R, \qquad \theta = C\Theta, \qquad z = FZ,$$
$$\lambda = \frac{\sin\alpha}{F\sqrt{(1-F^2\cos^2\alpha)}}, \qquad C = \frac{F(1-F^2\cos^2\alpha)}{\sin^2\alpha}, \tag{5.26}$$
$$\cos\alpha < \frac{1}{F}.$$

The tensor N^i_j is given by

$$N^i_j = (N^i_j)^+ + (N^i_j)^-,$$

where $(N^i_j)^+$ and $(N^i_j)^-$ correspond to the angles $+\alpha$ and $-\alpha$ of the two sets of fibers with scalars q^+ and q^-, respectively. Since $D = E = 0$, torsion and shearing are impossible and the remaining deformations being axially symmetric, it should be $q^+ = q^- = q$ and

$$[N^i_j] = q\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2\lambda^2C^2\sin^2\alpha & 0 \\ 0 & 0 & 2F^2\cos^2\alpha \end{bmatrix}. \tag{5.27}$$

The constitutive equation (2.6) is used, with N^i_j given by (5.27) and the equilibrium equations (2.1) take the form

$$-\frac{\partial p}{\partial r} + 2\lambda^2C^2\sin^2\alpha\frac{q}{r} + \frac{1}{r}\left[\beta_1\lambda^2(1-C^2) + \beta_{-1}\frac{1}{\lambda^2}\left(1-\frac{1}{C^2}\right)\right] = 0,$$
$$-\frac{\partial p}{\partial \theta} - 2\lambda^2C^2\sin^2\alpha\frac{\partial q}{\partial \theta} = 0, \tag{5.28}$$
$$-\frac{\partial p}{\partial z} - 2F^2\cos^2\alpha\frac{\partial q}{\partial z} = 0.$$

Imposing the condition $t^r_r = 0$,* p is found to be a constant

$$p = \beta_1\lambda^2 + \beta_{-1}\frac{1}{\lambda^2}. \tag{5.29}$$

Equation $(5.28)_1$ is then solved for q,

$$q = -\frac{1}{2\lambda^2C^2\sin^2\alpha}\left[\beta_1\lambda^2(1-C^2) + \beta_{-1}\frac{1}{\lambda^2}\left(1-\frac{1}{C^2}\right)\right]. \tag{5.30}$$

Equations $(5.28)_2$ and $(5.28)_3$ are automatically satisfied.

More specifically, for each particular type of deformation one draws the following conclusions.

(a) *Inflation, eversion, torsion* and *shearing* are impossible.

(b) *Extension* ($C = 1, D = E = 0$)

Equations (5.26) reduce to

$$r = \lambda R, \qquad \theta = \Theta, \qquad z = FZ,$$
$$\sin^2\alpha = F(1-F^2\cos^2\alpha). \tag{5.31}$$

* This condition implies that t^θ_θ is also zero.

Equation $(5.31)_4$ can be solved for F and hence λ can be obtained from $(5.26)_4$.

From (5.30) it is found that $q = 0$.

For a complete tube, extension is produced by the traction $t_{\langle zz\rangle}$, the only nonzero stress here, given by

$$t_{\langle zz\rangle} = \beta_1(F^2-\lambda^2)+\beta_{-1}\left(\frac{1}{F^2}-\frac{1}{\lambda^2}\right).$$

The reinforcement here does not strengthen the material at all i.e. the universal solution of the unconstrained material has been recovered. The deformed fibers have helicoidal shape but with different angle.

(c) *Bending and extension* $(D = E = 0)$

The deformation is defined by (5.26) and is produced by an assigned traction $t_{\langle zz\rangle}$ the only nonzero stress, given by

$$t_{\langle zz\rangle} = -p+\beta_1F^2+\beta_{-1}\frac{1}{F^2}-2F^2q\cos^2\alpha. \tag{5.32}$$

Equation (5.32) is used to compute the constant F.

The contribution of the reinforcement is here significant and q is positive for $C > 1$. The deformed fibers have helicoidal shape but the inclination angle is different than α.

6. CONCLUSIONS

All universal solutions determined here correspond to deformations which generate universal solutions even in the incompressible but otherwise unconstrained case. However, the addition of fiber reinforcement yields additional freedom which is reflected in the possibility of satisfying a greater number of boundary conditions.

The strengthening of the fiber-reinforced material is found significant for the following deformations: bending and shearing of a rectangular block with fibers along the Z direction; straightening and shearing of a sector of a circular tube with fibers along the Z direction; bending and extension of a sector of a circular tube radially reinforced; bending and extension and shearing of a sector of a circular tube with fibers along the Θ direction; inflation, eversion of a circular tube and bending and shearing of a sector of a circular tube with fibers along the Z direction; bending and extension of a sector of a circular tube with two sets of fibers of helical path.

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Абстракт—На основе метода инверсии получаются универсальные результаты для усиленных волокнами несжимаемых, изотропных, упругих материалов, под влиянием больших, упругих деформаций.

Исследуются следующие деформации: изгиб и сдвиг прямоугольного бруса; выпрямление и сдвиг сектора круглой трубы; вспучмвание, прощелкивание, удлинение, изгиб и сдвиг сектора круглой трубы.

Обсуждается вопрос укрепления материала вследствие армировки, признак напряженного состояния в волокнах и деформированная конфигурация волокон.